7.1 Integration by Parts

What do we know about integrating so far?

Antidifferentiation formulas:

$$\int x^n\,dx = \frac{x^{n+1}}{n+1} + C \quad (n \neq -1) \qquad \int \frac{1}{x}\,dx = \ln|x| + C$$

$$\int e^x\,dx = e^x + C \qquad \int b^x\,dx = \frac{b^x}{\ln b} + C$$

$$\int \sin x\,dx = -\cos x + C \qquad \int \cos x\,dx = \sin x + C$$

$$\int \sec^2 x\,dx = \tan x + C \qquad \int \csc^2 x\,dx = -\cot x + C$$

$$\int \sec x \tan x\,dx = \sec x + C \qquad \int \csc x \cot x\,dx = -\csc x + C$$

$$\int \sinh x\,dx = \cosh x + C \qquad \int \cosh x\,dx = \sinh x + C$$

$$\int \tan x\,dx = \ln|\sec x| + C \qquad \int \cot x\,dx = \ln|\sin x| + C$$

$$\int \frac{1}{x^2+a^2}\,dx = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) + C \qquad \int \frac{1}{\sqrt{a^2-x^2}}\,dx = \sin^{-1}\left(\frac{x}{a}\right) + C, \quad a > 0$$

Integration by substitution "undoes" differentiation by chain rule:

$$\textit{differentiation with}\ \_\_\_\_\_\_\_\_\_\_ \rightarrow$$

$$\frac{d}{dx}\sin\left(x^2\right) \quad = \_\_\_\_\_\_\_\_\_\_$$

$$\leftarrow \text{int}\,\textit{egration with}\ \_\_\_\_\_\_\_\_\_\_$$

$$\sin\left(x^2\right) + C = \int \qquad\qquad dx$$

How can we "undo" differentiation by product rule?:

$$\text{differentiation with} \underline{\hspace{3cm}} \rightarrow$$

$$\frac{d}{dx}\left[x^2e^x\right] \;\; = \underline{\hspace{3cm}}$$

$$\leftarrow \text{int}\,\textit{egration with} \underline{\hspace{3cm}}$$

$$x^2e^x + C = \int \qquad\qquad\qquad\qquad dx$$

Development of Integration by Parts:

Recall that if f, g are differentiable,

$$\frac{d}{dx}\left[fg\right] =$$

So integrating with respect to x,

$$fg =$$

Then,

$$\int fg'\,dx =$$

Example: (do once)

$$\int x\cos x\,dx =$$

Simplifying notation:

$$\int fg'\,dx = fg - \int f'g\,dx \quad \text{if we let} \quad \boxed{\begin{array}{ll} u = f(x) & dv = g'(x)dx \\ du = f'(x)dx & v = g(x) \end{array}}$$

The method of Integration by parts is written:

$$\boxed{\int u\,dv = \qquad\qquad\qquad\qquad}$$

Example (do twice):

$$\int x\cos x\,dx =$$

Tip: Choose u and dv such that

$$\int x \sec^2 x \, dx =$$

Example (twice):

$$\int x^2 e^x \, dx =$$

Example:

$$\int x^2 \ln x \, dx =$$

Example (unexpected, not product):

$$\int \ln x \, dx =$$

Example (unexpected, not product):

$$\int \sin^{-1}(x)\,dx =$$

Example (“circular”):

$$\int e^x \cos x\,dx =$$

ILATE

Example (combine with u-sub):

$$\int x^5 \cos\left(x^3\right) dx =$$

Example (definite integral): Watch notation

$$\int_0^1 x e^x \, dx$$

7.2 Trigonometric Integrals

In this section, we will consider strategies for finding integrals of powers and combinations of trigonometric functions. This is not a new method but a utilization of u-substitution, integration by parts, and ________________________

Integrals of trig functions

$$\int \sin x\,dx = \underline{\hspace{4cm}} \qquad \int \csc x\,dx = \underline{\hspace{4cm}}$$

$$\int \cos x\,dx = \underline{\hspace{4cm}} \qquad \int \sec x\,dx = \underline{\hspace{4cm}}$$

$$\int \tan x\,dx = \underline{\hspace{4cm}} \qquad \int \cot x\,dx = \underline{\hspace{4cm}}$$

Development $\int \sec x\,dx$

$$\int \sec x\,dx = \int \sec x \frac{\qquad\qquad}{\qquad\qquad}\,dx$$

Similarly, we can show $\int \csc x\,dx = \ln\left|\csc x - \cot x\right| + C$

Integrals of type: $\int \cos^n x dx$ $\quad \int \sin^n x dx$

Use trig identities:

$\cos^2 x + \sin^2 x = 1$	$\frac{d}{dx}[\sin x] =$ ____________	$\frac{d}{dx}[\cos x] =$ ____________
$\cos^2 x =$ ________________	$\sin^2 x =$ ________________	
$\sin 2x =$ ________________	$\sin x \cos x =$ ____________	

Examples:

$\int \cos^2 x dx =$ ________________ $\qquad$ $\int \sin^2 x dx =$ ________________

$\int \cos^2 (3x)\, dx$

$$\int \sin^3(x)\,dx$$

$$\int \cos^4(x)\,dx$$

Power reducing:

Integration by parts:

See example 6 page 515 for reduction formula derivation $\int \sin^n(x)\,dx$

Integrals of type: $\int \sin^m x \cos^n x dx$ (book has formulas page 521…better to *think*)

$$\int \sin^2(x)\cos^3(x)\,dx$$

$$\int \frac{\sin^4 x}{\cos^2 x}\,dx$$

Integrals of type: $\int \tan^m x \sec^n x dx$ (book has formulas page 521…better to *think*). $\int \cot^m x \csc^n x dx$ similar logic

Use trig facts:

$1+\tan^2 x = \sec^2 x$ $\frac{d}{dx}\left[\tan x\right] =$ ____________ $\frac{d}{dx}\left[\sec x\right] =$ ______________________

$\int \tan^2 x \sec^4 x\, dx$

$\int \tan^5 x \sec^7 x\, dx$

$$\int \sec^3 x\,dx$$

$$\int \tan^5 x\,dx$$

Many other types of trig integrals. Apply identities, u-sub, integration by parts

$$\sin A\ \cos B = \tfrac{1}{2}[\sin(A - B) + \sin(A + B)]$$

$$\sin A\ \sin B = \tfrac{1}{2}[\cos(A - B) - \cos(A + B)]$$

$$\cos A\ \cos B = \tfrac{1}{2}[\cos(A - B) + \cos(A + B)]$$

$$\int \sin 3x \sin 4x \, dx$$

$$\int \tan^2 x \sin x dx$$

7.3 Trigonometric Substitution

A technique for integrating expressions containing $a^2 - x^2, \quad x^2 - a^2, \quad a^2 + x^2$

Consider the expression $\sqrt{a^2 - x^2}$. Suppose you make the substitution $x = a\sin\theta, \quad -\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$:

$\sqrt{a^2 - x^2}$ =

We will find that expressions containing $a^2 - x^2$ can often be written in an algebraically simpler form by making the substitution $x = a\sin\theta, \quad -\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$ or equivalently $\theta = \sin^{-1}\left(\frac{x}{a}\right)$. This type of "inverse substitution" is called trigonometric substitution.

Example:

$$\int \sqrt{9 - x^2}\, dx$$

$$\int_{-3}^{3} \sqrt{9 - x^2}\, dx$$

Example: $a^2 - x^2 \Rightarrow$ Let $x = a\sin\theta, \quad -\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$

$$\int \frac{x^2}{(25 - x^2)^{3/2}} dx$$

$$\int \sqrt{1 - 16x^2}\, dx$$

$$\int_0^1 \frac{1}{\sqrt{4 - x^2}} dx$$

$$\int 2x\sqrt{1 - x^2}\, dx$$

For expressions containing a^2+x^2, a substitution of $x=a\tan\theta,\quad -\frac{\pi}{2}<\theta<\frac{\pi}{2}$, or equivalently $\theta=\sin^{-1}\left(\frac{x}{a}\right)$ will often provide algebraic simplification. Try it with the expression:

$$\sqrt{a^2+x^2}$$

Example:

$$\int\frac{dx}{(3+x^2)^{3/2}}$$

$$\int\frac{\sqrt{1+4x^2}}{x^4}dx$$

For expressions containing $x^2 - a^2$, a substitution of $x = a\sec\theta, \quad 0 \le \theta < \frac{\pi}{2} \cup \pi \le \theta < \frac{3\pi}{2}$, or equivalently $\theta = \sec^{-1}\left(\frac{x}{a}\right)$ will often provide algebraic simplification. Try it with the expression:

$$\sqrt{x^2 - a^2}$$

Example:

$$\int \frac{1}{\sqrt{x^2 - 16}} dx$$

Integrals can sometimes be put into the form needed for trig substitution by <u>completing the square</u>.

$$\int \frac{x}{\sqrt{3 - 2x - x^2}} dx$$

7.4 Integration of Rational Functions by Partial Fraction Decomposition

Algebra Review: $\frac{2}{x+1}+\frac{3}{x-3}=$

Consider the integral:

$$\int \frac{5x-3}{x^2-2x-3}\,dx$$

The process of breaking a fraction into simpler terms is a method you may have learned in precalculus called ____________________

Algebra Review: Partial Fraction Decomposition

Given the rational function. $R(x)=\frac{P(x)}{Q(x)}$ where $P(x), Q(x)$ are polynomials with no common factors, and $\deg P(x) < \deg Q(x)$ with $Q(x)$ expressed as a product of:

linear factors ___________________, repeated linear factors _________________________,

irreducible quadratic factors _______________with $b^2-4ac<0$ and repeated irreducible quadratic factors ____________________.,
the initial breakdown of the factors depend on the types of factors in the denominator $Q(x)$

Example: Identify each of the factors: $(3x-2)(x-1)^2x^3(x^2+4)(x^2+x+1)^3(x^2-x-2)$

Algebra Review: Partial Fraction Decomposition continued

Initial Breakdown:

For each distinct linear factor $ax+b$ in , there is a term of the form $\frac{}{ax+b}$ in the breakdown.

For each distinct irreducible quadratic factor ax^2+bx+c in , there is a term of the form $\frac{}{ax^2+bx+c}$ in the breakdown.

For each repeated linear factor $(ax+b)^n$ in , there are terms of the form $\frac{A_1}{ax+b}, \frac{A_2}{(ax+b)^2}, \frac{A_3}{(ax+b)^3} \dots \frac{A_n}{(ax+b)^n}$ in the breakdown.

For each repeated irr .quad factor $(ax^2+bx+c)^n$ in , there are terms of the form $\frac{A_1x+B_1}{ax^2+bx+c}, \frac{A_2x+B_2}{(ax^2+bx+c)^2}, \dots \frac{A_nx+B_n}{(ax^2+bx+c)^n}$ in the breakdown.

Examples of initial breakdown:

Finding the numerator coefficients: Generate a system of equations by "Equating Coefficients", and/or "choosing helpful x values"

Example: $\frac{5x-3}{x^2-2x-3}$

Examples: Decompositions done on the algebra review video.

$$\int \frac{1}{x(x-1)^2}\,dx$$

$$\int \frac{1}{x(x-1)^2}\,dx = \int \left(\frac{1}{x} - \frac{1}{x-1} + \frac{1}{(x-1)^2} \right) dx$$

$$\int \frac{x^2}{(x-1)(x^2+1)}\,dx$$

$$\int \frac{x^2}{(x-1)(x^2+1)}\,dx = \int \left(\frac{1}{2(x-1)} + \frac{x+1}{2(x^2+1)} \right) dx$$

Example:

$$\int \frac{2x^3+x+7}{(x^2+1)^2}\,dx$$

$$\int \frac{2x^3+x+7}{(x^2+1)^2}\,dx = \int \left(\frac{2x}{x^2+1} + \frac{-x+7}{(x^2+1)^2} \right) dx$$

Improper Fraction $\int \frac{x^3+1}{x^2(x-1)}\,dx$

$$\int \left(1 - \frac{1}{x} - \frac{1}{x^2} + \frac{2}{x-1} \right) dx$$

Rationalizing Substitution

This is a very useful technique that is only mentioned in Example 9 and in problems 39-52.

If an integrand involves an expression of the form $\sqrt[n]{g(x)}$ then ___________________ may be a helpful, but not obvious, substitution.

Examples:

$$\int \sqrt{e^{2x}-1}\,dx$$

$$\int \frac{\sqrt{x}}{1+\sqrt[3]{x}}dx$$

7.5 Strategy for Integration.

Read and practice. The more you get, the better you will be at this.

Note, we still cannot find antiderivatives for everything. Common integrals that don't look too difficult, but cannot be done using elementary functions include:

$$\int e^{x^2}\,dx \qquad \int \sin(x^2)\,dx \qquad \int \frac{\sin x}{x}\,dx \qquad \int \sqrt{x^3+1}\,dx \qquad \int \frac{1}{\ln x}\,dx \qquad \int \cos\left(e^x\right)dx$$

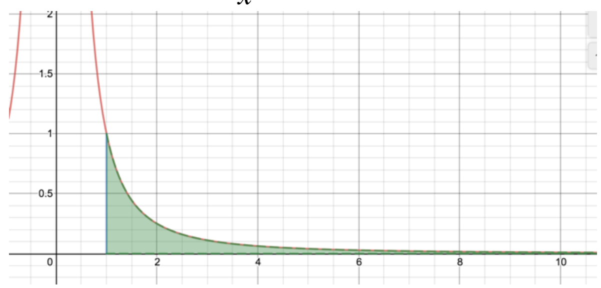
See also: Tables/Symbolic Integration/Numerical Methods 7.6, 7.7

7.8 Improper Integrals

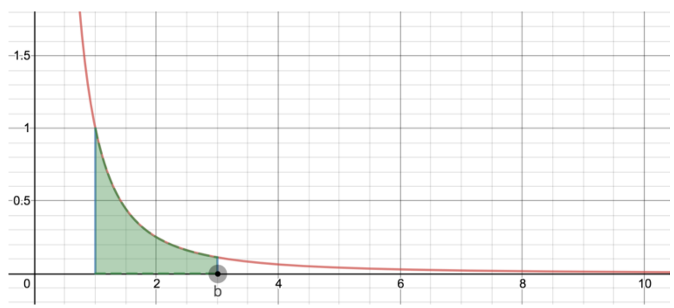
Up until now, we have defined $\int_a^b f(x)\,dx$ for $f(x)$ continuous, or bounded but with finitely many points of discontinuities, and a, b finite. Here we will consider the cases of infinite limits, and of f unbounded.

Improper integrals with infinite limits

Consider the following problem: Find the area under $f(x)=\dfrac{1}{x^2}$ in the first quadrant, $x \geq 1$.



Written as an integral, we might think of this as ____________________________ But what do we mean by this and how might we compute it? Let's consider a similar definite integral, $\int_1^b \dfrac{1}{x^2}\,dx$



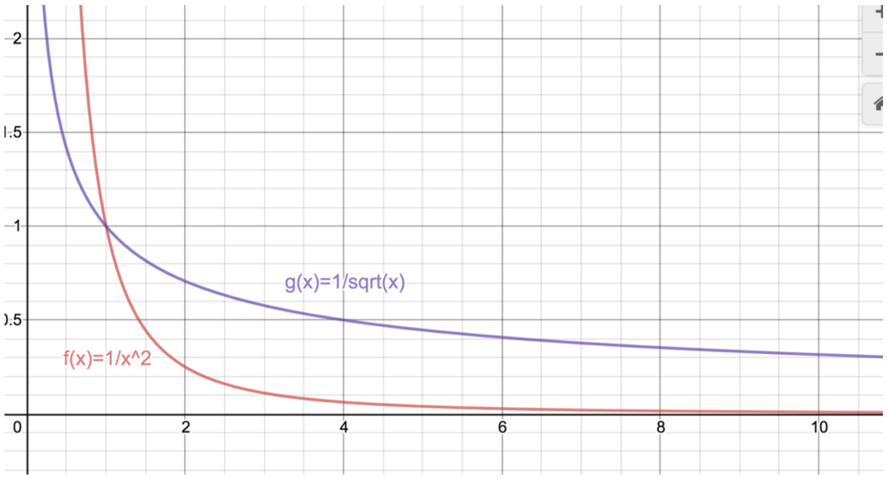
$\int_1^b \frac{1}{x^2}\,dx =$

Now let $b \to \infty$ (See Desmos illustration from 5B page)

$$\lim_{b\to\infty}\int_1^b \frac{1}{x^2}\,dx = \lim_{b\to\infty}\left(\qquad \qquad \right) =$$

What happens if we similarly try to find the area under the curve $g(x) = \frac{1}{\sqrt{x}}$ in the first quadrant?

Why would this be?



1 Definition of an Improper Integral of Type 1

(a) If $\int_a^t f(x)\,dx$ exists for every number $t \geqslant a$, then

$$\int_a^\infty f(x)\,dx = \lim_{t\to\infty} \int_a^t f(x)\,dx$$

provided this limit exists (as a finite number).

(b) If $\int_t^b f(x)\,dx$ exists for every number $t \leqslant b$, then

$$\int_{-\infty}^b f(x)\,dx = \lim_{t\to-\infty} \int_t^b f(x)\,dx$$

provided this limit exists (as a finite number).

The improper integrals $\int_a^\infty f(x)\,dx$ and $\int_{-\infty}^b f(x)\,dx$ are called **convergent** if the corresponding limit exists and **divergent** if the limit does not exist.

(c) If both $\int_a^\infty f(x)\,dx$ and $\int_{-\infty}^a f(x)\,dx$ are convergent, then we define

$$\int_{-\infty}^\infty f(x)\,dx = \int_{-\infty}^a f(x)\,dx + \int_a^\infty f(x)\,dx$$

In part (c) any real number a can be used (see Exercise 76).

So for the above examples, we would say

$\displaystyle\int_1^\infty \frac{1}{x^2}\,dx$ is ________________________ and equals 1 while $\displaystyle\int_1^\infty \frac{1}{\sqrt{x}}\,dx$ is ___________________________________.

Examples: Improper integrals with infinite limits .

$$\int_{-\infty}^{\infty} xe^{-x^2}\, dx$$

$$\int_{-\infty}^{\infty} x^3\, dx$$

$$\int_{-\infty}^{0} \frac{1}{x^2 - 4x + 9}\, dx$$

Comparison Theorem Suppose that f and g are continuous functions with $f(x) \geqslant g(x) \geqslant 0$ for $x \geqslant a$.

(a) If $\int_a^\infty f(x)\,dx$ is convergent, then $\int_a^\infty g(x)\,dx$ is convergent.

(b) If $\int_a^\infty g(x)\,dx$ is divergent, then $\int_a^\infty f(x)\,dx$ is divergent.

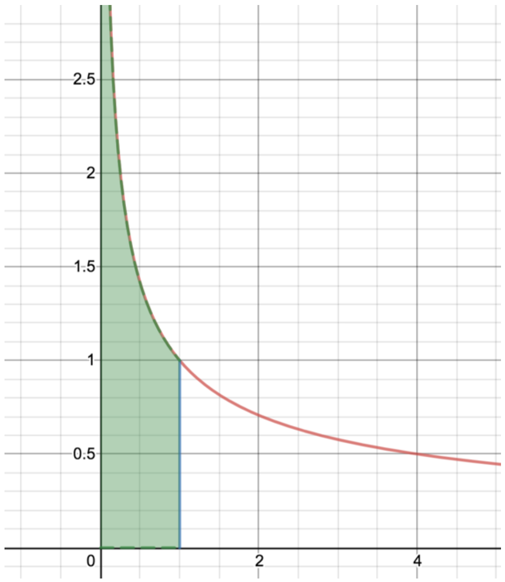
Example: Determine whether the following integrals converge or diverge.

$$\int_1^\infty \frac{\sin^2 x}{x^2}\,dx$$

$$\int_2^\infty \frac{x}{\sqrt{x^3 - x}}\,dx$$

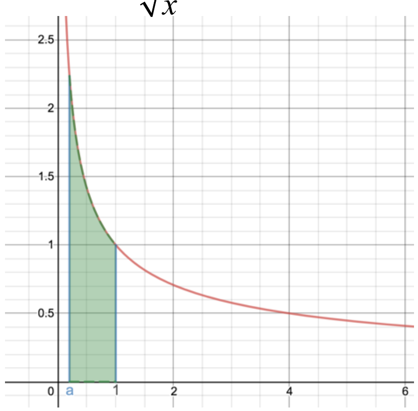
Improper integrals with f(x) unbounded

Consider the following problem: Find the area under $g(x)=\frac{1}{\sqrt{x}}$ over [0,1]



Written as an integral, we might think of this as ______________________ But what do we mean by this

given that $\frac{1}{\sqrt{x}}$ is not defined at x=0? Let's consider a similar definite integral, $\int_a^1 \frac{1}{\sqrt{x}}\,dx$.



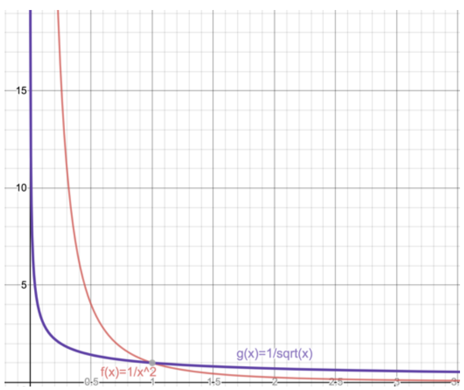
$$\int_a^1 \frac{1}{\sqrt{x}}\,dx =$$

Now let $a \to 0^+$ (why from the right?)

$$\lim_{a \to 0^+} \int_a^1 \frac{1}{\sqrt{x}}\,dx = \lim_{a \to 0^+}\left(\qquad\qquad\qquad \right) =$$

What happens if we similarly try to find the area under the curve $f(x) = \dfrac{1}{x^2}$ over [0,1].

Why would this be?



3 Definition of an Improper Integral of Type 2

(a) If f is continuous on $[a, b)$ and is discontinuous at b, then

$$\int_a^b f(x)\,dx = \lim_{t \to b^-} \int_a^t f(x)\,dx$$

if this limit exists (as a finite number).

(b) If f is continuous on $(a, b]$ and is discontinuous at a, then

$$\int_a^b f(x)\,dx = \lim_{t \to a^+} \int_t^b f(x)\,dx$$

if this limit exists (as a finite number).

The improper integral $\int_a^b f(x)\,dx$ is called **convergent** if the corresponding limit exists and **divergent** if the limit does not exist.

(c) If f has a discontinuity at c, where $a < c < b$, and both $\int_a^c f(x)\,dx$ and $\int_c^b f(x)\,dx$ are convergent, then we define

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$$

Example:

$$\int_0^2 \frac{1}{4 - x^2}\,dx$$

Example: Multiple “improprieties”

$$\int_0^{\infty} \frac{1}{3-x}\,dx$$